TOWARD A GENERAL THEORY OF TRANSMUTATION

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Abstract

A general construction of transmutation operators is developed for selfadjoint operators in Gelfand triples. Theorems regarding analyticity of generalized eigenfunctions and Paley-Wiener properties are proved.

1 Introduction

The idea of transmutation operator (or transformation operator) B such that BP = QB for P and Q ordinary differential operators goes back to Gelfand, Levitan, Marchenko, Naimark, et. al. in the early 1950's (cf. [23;25;28]). It was picked up again by Delsarte and Lions, who established some fundamental ideas (cf. [26;31]), and subsequently it was developed in many directions (see e.g. [2;10-14;17;22;31]). In this article we indicate some constructions of a general nature which will be further enhanced in subsequent papers. We develop the theory via selfadjoint operators in Gelfand triples and give some constructions of transmutation operators with various domains. Then various properties such as analyticity of generalized eigenfunctions and Paley-Wiener properties are discussed, with results of various kinds.

2 Background

A typical background situation involves $P = -D^2$ and $Q = -D^2 + q$ (q real) on $[0, \infty)$, $D \sim \partial_x$ where e.g. $q \in C^0[0, \infty)$ and $\int_{-\infty}^{\infty}(1 + x^2)|q|dx < \infty$. This is a typical

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inverse scattering situation and we denote by $\phi = Coskx$ and ψ the generalized eiqenfunctions satisfying

$$P\phi = k^2\phi;\ Q\psi = k^2\psi;\ \phi(0,k) = \psi(0,k) = 1;\ \phi'(0,k) = \psi'(0,k) = 0 \tag{2.1}$$

Here we will also write $\lambda = k^2$ and $k = \sqrt{\lambda}$, depending on context, with abuse of notation such as $\psi(x,k) \sim \psi(x,\lambda)$ when no confusion can arise, and one notes that $\phi \sim \phi_k^P$ with $\psi \sim \phi_k^Q$ in the notation of [11-14]. More general initial conditions $h\psi(0,k) - \psi'(0,k) = 0$ can also be envisioned. Then (cf. [11;12;28] for details) one can produce by PDE techniques or by Paley-Wiener theory a triangular kernel K(x,y) such that

$$\psi(x,\lambda) = \phi(x,\lambda) + \int_0^x K(x,t)\phi(t,\lambda)dt = (B\phi(\cdot,\lambda)) \tag{2.2}$$

This can be written $\psi(x,\lambda) = (B\phi(\cdot,\lambda)) =< \beta(x,t), \phi(t,\lambda) >$ for $\beta(x,t) = \delta(x-t) + K(x,t)$ and we will write, for suitable f,

$$(B^*f)(t) =< \beta(x,t), f(x) >= f(t) + \int_t^\infty K(x,t)f(x)dx \tag{2.3}$$

Further in the same way one can transmute in the opposite direction via

$$\phi(x,\lambda) = \psi(x,\lambda) + \int_0^x L(x,t)\psi(t,\lambda)dt = (\mathcal{B}\psi(\cdot,\lambda)) \tag{2.4}$$

Thus we will write BP = QB and $\mathcal{B}Q = P\mathcal{B}$ with $\gamma(x,y) = \delta(x-y) + L(x,y)$.

We emphasize that $\phi,\ \psi \notin L_x^2$ and we are not at the moment dealing with an L^2 theory; the brackets $<, >$ denote suitable distribution pairings. Assume now that Q has only continuous spectrum (P does of course) and consider transforms (for suitable f)

$$\mathcal{P}f(k) = \mathcal{C}f(k) = \int_0^\infty Coskx\, f(x)dx; \tag{2.5}$$

$$\mathcal{P}^{-1}F(x) = \frac{2}{\pi}\int_0^\infty Coskx\, F(k)dk;\ \mathcal{Q}f(k) = \int_0^\infty f(x)\psi(x,k)dx$$

Next it is shown in [11;12;27] for example that there is a generalized spectral function $R^Q \in Z'$ and a Parseval formula

$$< f, g >=< R^Q, \mathcal{Q}f\mathcal{Q}g > \quad (2.6$$

for functions $f, g \in K^2 = \{f \in L^2(0,\infty)$ with compact support$\}$. Here $\mathcal{P}K^2 = \mathcal{C}K^2 = \cup\mathcal{C}K^2(\sigma)$ for $\mathcal{C}K^2(\sigma) = \{$even entire $\hat{f}(k) = \mathcal{P}f$ with $\hat{f} \in L^2$ for k real

and $|\hat{f}(k)| \leq cexp(\sigma|Im(k)|)$ via $f \in K^2(\sigma)$ or $supp(f) \subset [0,\sigma]\}$, and $Z = \cup Z(\sigma)$ for $Z(\sigma) = \{$even entire functions g(k) with $g \in L^1$ for k real and $|g(k)| \leq cexp(\sigma|Im(k)|)\}$. Z has a countable union topology as in [19] and Z' is its dual. From this one obtains an inversion

$$\mathcal{Q}^{-1}F(x) =< R^Q, F(k)\psi(x,k) > \tag{2.7}$$

This all leads to factorizations

$$\mathcal{P}B^* = \mathcal{Q};\ \mathcal{Q}\mathcal{B}^* = \mathcal{P} \tag{2.8}$$

Further let us define operators, when they make sense (here $R^P \sim \frac{2}{\pi}dk$)

$$\tilde{\mathcal{Q}}F(x) =< F(k)\psi(x,k), R^P >; \tilde{\mathcal{P}}F(x) =< F(k)\phi(x,k), R^Q > \tag{2.9}$$

Then e.g. via formal representations

$$\beta(y,x) =< \phi(x,k)\psi(y,k), R^P >; \gamma(x,y) =< \phi(x,k)\psi(y,k), R^Q > \tag{2.10}$$

one can write

$$B = \tilde{\mathcal{Q}}\mathcal{P};\ \mathcal{B} = \tilde{\mathcal{P}}\mathcal{Q};\ B^* = \mathcal{P}^{-1}\mathcal{Q};\ \mathcal{B}^* = \mathcal{Q}^{-1}\mathcal{P} \tag{2.11}$$

Such formulas were applied to many operators P and Q in [11;12;14], involving both singular and nonsingular situations, and many explicit formulas for kernels etc. were obtained in terms of special functions. It was most often the case the the generalized spectral functions R^Q and R^P were in fact measures $d\Gamma_Q$ and $d\Gamma_P$ in which case one can rewrite the spectral pairings etc. as integrals. In particular (2.6) becomes

$$\int fg\,dx = \int \mathcal{Q}f\,\mathcal{Q}g\,d\Gamma_Q \tag{2.12}$$

Further the measures $d\Gamma_P$, $d\Gamma_Q$ were frequently absolutely continuous with say $d\Gamma_P = \gamma_P dk$ and this will set the stage for our presentation in section 3.

3 Transmutation for certain selfadjoint operators

The theory described in section 2 was based on differential operators $Q = -D^2 + q$ and the associated Paley-Wiener theory for example. We want to deal now with a more general situation where less is assumed a priori and which will include the type of situation described in section 2. Thus take a densely defined selfadjoint operator Q in $L^2(dM_Q) = L^2_Q$ with a simple spectrum (cf. [1]). This entails no basic loss

in generality since for finite multiplicity one could decompose Q as a direct sum of operators with simple spectrum. For $\lambda \in sp(Q) = \sigma_Q$ there exists a sequence $f_n \in D(Q) \subset L^2_Q$ such that $\|f_n\| = 1$ and $lim\,\|Qf_n - \lambda f_n\| = 0$. If $f_n \to f$ in L^2_Q then $\lambda \in \sigma^d_Q$ (discrete spectrum) and $Qf = \lambda f$ with $f \in L^2_Q$. If f_n does not converge then $\lambda \in \sigma^c_Q$ (continuous spectrum). We imagine then that $L^2_Q \hookrightarrow \Phi'$ with compact embedding (e.g. think of Gelfand triples or rigged Hilbert spaces as in [3;4;13;18-20;28]) and then $f_n \to F \in \Phi'$. Thus we will have a solution $F \sim \psi(x,\lambda) \in \Phi'$ of $Q\psi = \lambda\psi$ for $\lambda \in \sigma_Q$. As to constructing such Φ one recalls (cf. [1]) that given a selfadjoint operator Q in a Hilbert space H with simple spectrum there is a vector $h \in D(Q) \subset H$ such that $Q^k h$ is defined for all k and the linear envelope $\Phi = \{Q^k h\}$ is dense in H. Put on Φ if possible a topology such that $i : \Phi \hookrightarrow H$ is compact and embed H in the antidual Φ' via $\phi \to (\phi, h) =< L_h, \phi >$ (for convenience from now on we will think of real Hilbert spaces without loss of generality -cf. [12;14]). Then $\Phi \subset D(Q)$ and $Q : \Phi \to \Phi$ is continuous (variations on this are indicated below). Note $H_n = \{\sum_{|k|\leq n} a_k Q^k h\}$ is finite dimensional, hence nuclear, and $\Phi = \overrightarrow{lim}\, H_n$ is nuclear with $\Phi \hookrightarrow H$. However $i : \Phi \to H$ is not a priori compact or Hilbert-Schmidt without further hypotheses. The theory of rigged spaces (cf. [3;19;20;29]) then provides a measure $d\Gamma_Q$ with ($f \in \Phi$)

$$\mathcal{Q}f(\lambda) = \hat{f}(\lambda) =< f(x), \psi(x,\lambda); \tag{3.1}$$

$$f(x) = \int_{-\infty}^{\infty} \mathcal{Q}f(\lambda)\psi(x,\lambda)d\Gamma_Q(\lambda); \|f\|^2 = \int_{-\infty}^{\infty} |\mathcal{Q}f(\lambda)|^2 d\Gamma_Q(\lambda)$$

Thus $\mathcal{Q} : \Phi \to L^2(d\Gamma_Q)$ is continuous and this can be extended to an isometry $\bar{\mathcal{Q}} : H \to L^2(d\Gamma_Q)$, which we usually denote again by $\mathcal{Q}$. We observe now that given the spectral measure $d\Gamma_Q$ and $\psi(x,\lambda) \in$ (some $\hat{\Phi}'$) with $H \sim L^2_Q$ but no $\hat{\Phi}$ in sight, it would be natural to expect $\mathcal{S}_\lambda \subset L^2(d\Gamma_Q)$ ($\mathcal{S}$ = Schwartz space). Then from the property $Q\psi = \lambda\psi$ one has for $F \in \mathcal{S}$ and $f(x) = \mathcal{Q}^{-1}F = \int F(\lambda)\psi(x,\lambda)d\Gamma_Q$,

$$Q^n f = \int F\, Q^n \psi\, d\Gamma_Q = \int (\lambda^n F)\psi d\Gamma_Q \tag{3.2}$$

which is well defined since $\lambda^n F \in \mathcal{S}$ again. It is not clear however how $\tilde{\Phi} = \mathcal{Q}^{-1}\mathcal{S}$ is related to Φ for Φ constructed above or to a putative $\hat{\Phi}$. In any case $\tilde{\Phi}$ will be a perfect space (bounded sets are relatively compact) with the topology defined via seminorms

$$\|\tilde{\phi}\|_p = |\mathcal{Q}\tilde{\phi}|_p = sup_{q\leq p}|(1+\lambda^{2p})\partial^q_\lambda(\mathcal{Q}\tilde{\phi})(\lambda)| \tag{3.3}$$

(i.e. with the topology induced by $\mathcal{S}$ and $\mathcal{Q}$). Further $\tilde{\Phi} \hookrightarrow H$ with compact embedding and $\tilde{\Phi} \hookrightarrow H \hookrightarrow \tilde{\Phi}'$ with $Q\tilde{\Phi} \subset \tilde{\Phi}$. Hence one knows there exists a

generalized eigenfunction $\tilde{\psi}(x,\lambda) \in \tilde{\Phi}'$ and a measure $d\tilde{\Gamma}_Q$ with formulas (3.1) for $\tilde{\mathcal{Q}}f(\lambda) =< f(x), \tilde{\psi}(x,\lambda) >$. One has $\tilde{\mathcal{Q}}\tilde{\Phi} = \mathcal{S} \subset L^2(\tilde{\Gamma}_Q)$ but generally one may not have $\psi(x,\lambda) \in \tilde{\Phi}'$. This is now circumvented by transferring the theory of Q to $\tilde{\Phi} \hookrightarrow H \hookrightarrow \tilde{\Phi}'$ and using $\tilde{\psi}(x,\lambda)$ in place of $\psi(x,\lambda)$. Consequently, given a selfadjoint Q in H with simple spectrum, once we have a generalized eigenfunction $\psi(x,\lambda)$ with formulas (3.1) we can produce a Gelfand triple $\tilde{\Phi} \hookrightarrow H \hookrightarrow \tilde{\Phi}'$ with an isomorphic theory. This leads us to work with the class of operators (s.a. $\sim$ selfadjoint)

$$\mathcal{A} = \{s.a.\ Q\ in\ H \sim L^2_Q\ with\ (3.1),\ \Phi \hookrightarrow H \hookrightarrow \Phi',\ \Phi \subset D(Q)\ dense,\ Q\Phi \subset \Phi\} \tag{3.4}$$

where Φ is to be dense in D(Q) with graph norm and in H.

Now given Q_1, $Q_2 \in \mathcal{A}$ (based on $\Phi_i \hookrightarrow H_i \hookrightarrow \Phi_i'$) let us assume first $\sigma_1 = \sigma_2$ for simplicity and write for $f \in \Phi_i$

$$\hat{f}_i(\lambda) =< f(x), \psi_i(x,\lambda) >;\ f(x) = \int \hat{f}_i(\lambda)\psi_i(x,\lambda)d\Gamma_i(\lambda) \tag{3.5}$$

(recall we are using real L^2 spaces for convenience - the corresponding results for complex spaces will follow as indicated in [12;14]). Denote by $\mathcal{Q}_i$ the maps indicated in (3.1) so $\mathcal{Q}_i$ extends to an isometry $\mathcal{Q}_i : H_i \to L^2(d\Gamma_i)$ ($H_i \sim L^2_{Q_i} = L^2(dM_i)$). Define now a map

$$\mathbf{Q} : L^2(d\Gamma_1) \to L^2(d\Gamma_2) :\ \hat{f}_1(\lambda) \to \mathbf{Q}\hat{f}_1(\lambda) = \hat{f}_2(\lambda) \tag{3.6}$$

Here $f \in H_1 \cap H_2$ and $D(\mathbf{Q}) = \{F \in L^2(d\Gamma_1);\ \mathcal{Q}_1^{-1}F \in H_1 \cap H_2\}$ with

$$\begin{array}{ccc} \hat{f}_1 & \stackrel{\mathbf{Q}}{\to} & \mathbf{Q}\hat{f}_1 = \hat{f}_2 \\ \mathcal{Q}_1 \uparrow & \mathcal{Q}_2 \nearrow & \Uparrow \mathcal{Q}_1 \\ f \in H_1 \cap H_2 & \stackrel{V}{\Rightarrow} & Vf \in H_1 \end{array}$$

We will assume here that $H_1 \cap H_2$ is dense in H_i. The double arrows indicate a formal relation to the machinery of section 1 where $V = \mathcal{Q}_1^{-1}\mathcal{Q}_2 \sim B^*$ in (2.10). Here we think of transmutations $B : Q_1 \to Q_2$, $BQ_1 = Q_2B$, acting on $f \in D(Q_1)$ with $Bf \in D(Q_2)$. It is important to notice that $\mathbf{Q}$ is not a multiplication operator in general. Note that, with suitable definition of domains, $B : Q_1 \to Q_2$ is equivalent to $Q_1B^* = Q_1^*B^* = B^*Q_2^* = B^*Q_2$, or $B^* : Q_2 \to Q_1$. Similarly $\mathcal{B} \sim B^{-1}$ satisfies $Q_1B^{-1} = B^{-1}Q_2$, so $\mathcal{B} : Q_2 \to Q_1$. We point out in passing however that B^* and B^{-1} have opposite triangularities (cf. (2.3), (2.5), etc.). Now we will prove (note a priori $\Phi_1 \cap \Phi_2$ could be $\{0\}$ - cf. also Theorem 4.11)

THEOREM 3.1. The operator V defined by

$$\mathcal{Q}_2 f(\lambda) = \mathcal{Q}_1(Vf)(\lambda) \tag{3.7}$$

for $f \in A$ (below - A dense in $D(Q_2) \cap H_1$) will satisfy $VQ_2 f = Q_1 V f$.

Proof: Let $A = \{f \in H_1 \cap H_2;\ (1+|\lambda|)\mathcal{Q}_2 f(\lambda) \in L^2(d\Gamma_1) \cap L^2(d\Gamma_2)\}$. Note A will be dense under our assumptions since the space of such $(1+|\lambda|)\mathcal{Q}_2 f$ will be dense in $L^2(d\Gamma_1) \cap L^2(d\Gamma_2)$. For $f \in A$ one defines Vf via (3.7) and it follows via the diagram that

$$\lambda \mathcal{Q}_2 f(\lambda) = \lambda \mathcal{Q}_1(Vf)(\lambda) \tag{3.8}$$

Now from $Q_i \psi_i = \lambda \psi_i$ one can say that for $f \in \Phi_i \subset D(Q_i)$

$$\lambda \mathcal{Q}_i f(\lambda) = < f(x), \lambda \psi_i(x,\lambda) >= \tag{3.9}$$

$$= < f(x), Q_i \psi_i(x,\lambda) >=< Q_i f, \psi_i(x,\lambda) >= \mathcal{Q}_i(Q_i f)$$

and this extends to $f \in D(Q_i)$. Now the left side of (3.8) becomes (for $f \in A$), $\mathcal{Q}_2(Q_2 f) = \mathcal{Q}_1(VQ_2 f)$ (by (3.7)) and the right side is $\mathcal{Q}_1(Q_1 V f)$, provided $Vf \in D(Q_1)$. But in fact, writing $\mathcal{Q}_i \sim \bar{\mathcal{Q}}_i$ in H_i via scalar products $(\ ,\ )_i$, i.e. $\bar{\mathcal{Q}}_i = (h, \psi_i)_i$, the equation $\mathcal{Q}_1(VQ_2 f) = \lambda \mathcal{Q}_1(Vf)$ can be expressed as

$$\mathcal{Q}_1(VQ_2 f) = (VQ_2 f, \psi_1(x,\lambda))_1 = (Vf, \lambda\psi_1)_1 = (Vf, Q_1 \psi_1)_1 \tag{3.10}$$

(we emphasize $\psi_i \notin H_i$ but in expressing the action of $\bar{\mathcal{Q}}_i$ we will use the $(\ ,\ )_i$ notation - see section 4 for more detail). Now elements $g \in D(Q_1)$ can be expressed formally via $g = \int (\mathcal{Q}_1 g)\psi_1(x,\lambda) d\Gamma_1$ so, with a little argument by approximation

$$(VQ_2 f, g)_1 = (VQ_2 f, \int \hat{g}_1 \psi_1 d\Gamma_1)_1 \tag{3.11}$$

$$= \int \hat{g}_1 (VQ_2 f, \psi_1)_1 d\Gamma_1 = \int \hat{g}_1 (Vf, \lambda\psi_1)_1 d\Gamma_1 = (Vf, Q_1 g)_1$$

This means $g \to (Vf, Q_1 g)_1$ is continuous in the topology of H_1 so $Vf \in D(Q_1^*)$ and $VQ_2 f = Q_1 V f$ ($Q_1^* = Q_1$). A simpler argument can be based on $D(Q) = \{f : \hat{f}\ \textit{and}\ \lambda\hat{f} \in L^2(d\Gamma)\}$. **QED**

REMARK 3.3 One could make various assumptions regarding the Φ_i, $D(Q_i)$, etc to produce a cleaner looking theory. If e.g. $H_1 = H_2 = H$ then we have a traditional transmutation framework. We emphasize however that although $\mathcal{Q}_1 : H \to L^2(d\Gamma_1)$ and $\mathcal{Q}_2 : H \to L^2(d\Gamma_2)$ are isometries, we cannot say that $V =$

$\mathcal{Q}_1^{-1}\mathcal{Q}_2 : D(Q_2) \to D(Q_1)$ will extend to a bounded operator in H. This is simply because the injection $i : L = L^2(d\Gamma_1) \cap L^2(d\Gamma_2) \to L^2(d\Gamma_1)$ may not be continuous when L has the $L^2(d\Gamma_2)$ topology. Thus in general V will not be a bounded operator. However $\mathbf{Q} = \mathcal{Q}_2\mathcal{Q}_1^{-1} : L^2(d\Gamma_1) \to L^2(d\Gamma_2)$ will be continuous (recall the $\mathcal{Q}_i$ or $\bar{\mathcal{Q}}_i$ are isometries). Further if one has e.g. $d\Gamma_1 = \gamma_1 d\lambda,\ d\Gamma_2 = \gamma_2 d\lambda$, with $|\gamma_1/\gamma_2| \leq M < \infty$, then i is bounded and the theorem will apply for $f \in D(Q_2)$ (V will be bounded in this situation).

4 Analysis of Q

We recall a theorem of Levitan [24] which states that every continuous linear operator in a space of analytic functions $\mathcal{H}$ is locally a linear differential operator of (possibly) infinite order. Here the appropriate topology is that of uniform convergence on compact sets, i.e. $F_n \to F$ means that for any fixed compact K, $sup_{\lambda\in K}|F_n(\lambda) - F(\lambda)| \to 0$ (we write this as $F_n \stackrel{ucc}{\to} F$). Note that a proof is easily constructed via the Cauchy integral formula. Thus let $\mathbf{Q} : \mathcal{H}(\Omega) \stackrel{ucc}{\to} \mathcal{H}(\Omega)$ (Ω open) be continuous and linear and given a compact K with $z \in K$ let $K \subset C \subset \hat{K} \subset \Omega,\ \hat{K}$ compact (where C is a curve). Then $\mathbf{Q}F_n(z) = (1/2\pi i)\oint_C F_n(\xi)\mathbf{Q}(1/(\xi - z))d\xi$. It follows that $F_n \to 0$ uniformly on $\hat{K}$ implies $\mathbf{Q}F_n \to 0$ uniformly on K. Consequently, locally

$$\mathbf{Q}F(z) = \sum_0^\infty F^{(j)}(z)\frac{1}{2\pi i}\oint_C (\zeta - z)^j \mathbf{Q}(1/(\zeta - z))d\zeta = \sum_0^\infty a_j(z)\partial^j F(z)$$

Note that $\mathbf{Q}(1/(\xi - z))$ must be defined here so $1/(\xi - z)$ must be analytic for $\xi \in C,\ z \in K$.

Let now $W_i \subset H_i$ be the space of functions such that $\mathcal{Q}_i(W_i)$ is entire when extended to $\mathbf{C}$. One thinks here of C_0^∞ and Paley-Wiener (= PW) theorems for example so many examples exist where W_i will be dense (cf. [11;12]). Let us also assume for convenience that $dM_{Q_i} = dx$ (so $H_1 = H_2 = H$). A priori Φ_i and W_i may not have any nice relation but we note that the spaces W^Ω of [19] defined below will usually be available as dense subspaces of $L^2(d\Gamma_1) \cap L^2(d\Gamma_2)$ so that $\hat{\Phi}_i = \mathcal{Q}_i^{-1}W^\Omega \subset H_i$ and $\hat{\Phi}_i'$ could be used for a Gelfand triple (cf. remarks at the beginning of section 3). Hence we will assume $\Phi_i \subset W_i$ without loss of generality. As for W^Ω we define $W^\Omega = \{F\ entire;\ (1 + |\lambda|)^k|F(\lambda)| \leq c_k exp(\Omega(b|Im(\lambda)|))\}$ where $\Omega(y) = \int_0^y \xi(x)dx;\ \xi(x) \geq 0\ for\ \lambda > 0$. W^Ω is a countably normed space with seminorms $\|f\|_n = sup_\lambda(1 + |\lambda|)^n|F(\lambda)exp(-\Omega(b|Im(\lambda)|))$ and the convergence of sequences is defined by $F_n \stackrel{ucc}{\to} F$ with $(1+|\lambda|)^k|F_n(\lambda)| \leq c_k exp(\Omega(b|Im(\lambda)|))$ for all n,k. In the particular case when $\xi(t) = 1$, i.e., $\Omega(y) = y$, then $W^y = Z$, where Z is the space of entire functions of order one and finite type (i.e. exponential type) defined by

the family of seminorms $\|f\|_n^Z = sup_\lambda |F(\lambda)|(1+|\lambda|)^n exp(-a|Im(\lambda)|)$. Spaces of type W are known to be perfect and in the analysis to follow we shall make occasional use of such spaces (cf. [19] for more details). An operator A is continuous in the space W^Ω if it maps bounded sets into bounded sets or equivalently if $f_n \to 0$ implies $Af_n \to 0$. In the situations we consider with Lebesgue-Stieljtes measures $d\Gamma$, if we assume $d\Gamma = \gamma d\lambda$; $\gamma = O(|\lambda|^p)$, then $W^\Omega \subset L^2(d\Gamma)$ since $F(\lambda) = O(|\lambda|^{-n})$ for any $n > 0$ when $F \in W^\Omega$. Now one has

THEOREM 4.1. Assume $\mathbf{Q}$ can be extended to be a map $\mathbf{Q} : \mathcal{H} \to \mathcal{H}$, continuous in the ucc topology and $H_1 = H_2 = H$. If $f \in W_1$ then $\hat{f}_2(\lambda) = \mathbf{Q}\hat{f}_1(\lambda) \in \mathcal{H}$ so $f \in W_2$ and locally ($D \sim d/d\lambda$)

$$\mathbf{Q}\hat{f}_1(\lambda) = \hat{f}_2(\lambda) = \sum_0^\infty a_n(\lambda) D^n \hat{f}_1(\lambda) \tag{4.1}$$

It follows then, assuming $\Phi_i \subset W_i$ as discussed above, that $\psi_i(x, \cdot) \in \mathcal{H}$ weakly and as transform objects χ for $\mathcal{Q}$ acting via $f \to (f, \chi)$ for $f \in \Phi$ (or $f \in H$) one has (here ∂_λ^n refers to a weak or scalar derivative)

$$\psi_2(x, \lambda) = \sum_0^\infty a_n(\lambda) \partial_\lambda^n \psi_1(x, \lambda) \tag{4.2}$$

Proof: Here we say $\psi(x, \cdot) \in \mathcal{H}$ weakly if $\lambda \to < f(x), \psi(x, \lambda) > \in \mathcal{H}$ for any $f \in \Phi$ (recall $\psi \in \Phi'$). Now the formula (4.1) follows from [23] since for $f \in \Phi_1 \subset W_1$ we know $\hat{f}_1(\lambda) = < f(x), \psi_1(x, \lambda) > \in \mathcal{H}$ (so $\psi_1 \in \mathcal{H}$ weakly and similarly $\psi_2 \in \mathcal{H}$ weakly since $\Phi_2 \subset W_2$). The equation $\mathbf{Q}\hat{f}_1(\lambda) = \hat{f}_2(\lambda)$ can be written formally as

$$\hat{f}_2(\lambda) = \sum_0^\infty a_n(\lambda) \partial_\lambda^n < f(x), \psi_1(x, \lambda)_1 > \tag{4.3}$$

$$= < f(x), \sum a_n(\lambda) \partial_\lambda^n \psi_1(x, \lambda) >_1 = \int f(x) \psi_2(x, \lambda) dx$$

We do not know if $f \in \Phi_2$ ($f \in W_2$) and even if we assume $\Phi_1 \cap \Phi_2$ is dense in Φ_2, making the last term in (4.3) $< f(x), \psi_2(x, \lambda) >_2$, this forces a comparison of $<, >_1$ and $<, >_2$. Hence we want to use $H = H_1 = H_2$ as an identification space and write integral signs in (4.3) instead of $<, >_1$ (note however that we want to use $<, >_1$ first in order to differentiate in λ weakly). This implies that as transform objects χ acting via $f \to (f, \chi) = \Xi(f)$, $f \in \Phi$, we can make the identification (4.2). **QED**

There are a number of variations possible here (note also Proposition 4.6 below

which indicates that the continuity of $\mathbf{Q}$ is usually too strong). We remark first however that by the Riesz theorem one could also use H' as an identification space in Theorem 4.4. Thus $\psi \in \Phi'$ generates via $\mathcal{Q}$ a map $f \to (f, \chi_\psi) = \Xi_\psi(f)$ where $\chi_\psi \sim \psi$ for $f \in \Phi$ where $(f, \chi_\psi) =< f, \chi_\psi >=< f, \psi >$. Since a scalar product $\Xi_\psi(f) = (f, \chi_\psi) \sim< f, \text{Ł}_\psi >$ for $L_\psi \in H'$ we can identify $\psi \sim L_\psi \in H'$ etc. in (4.2). Another variation is to assume $\Phi = \Phi_1 \cap \Phi_2$ is dense in Φ_i with a suitable topology. Then one can also use Φ' as an identification space and write (4.2) as a genuine equation in weak derivatives in Φ' (cf. Theorem 4.7). Now regarding weak differentiability we recall that in the dual of a barreled LCS (= locally convex topological vector space) E the weak topology is equivalent to the topology of uniform convergence on precompact sets in E. If in addition bounded sets are relatively compact in E (i.e. E is a Montel space) then the weak topology in E' is equivalent to the strong topology. Further the strong dual of a Montel is Montel and evidently a barreled perfect space is Montel. Noting that strict inductive limits of barreled spaces are barreled one sees that Gelfand triples will often involve Montel spaces Φ and Φ'. Thus without great loss of generality we can assume $\Phi_i \subset H \subset \Phi_i'$ is a Gelfand-Montel triplet. Following [16;30] we have then (note stronger theorems and a comprehensive study of vector valued analytic functions are available in [21] but we include Corollaries 4.2 and 4.3 for completeness and to facilitate calculation)

COROLLARY 4.2. Let $\Phi_i \subset H \subset \Phi_i'$ be Gelfand-Montel triples and assume the other hypotheses of Theorem 4.1. Then the ψ_i are strongly analytic and the derivatives $\partial_\lambda^n \psi_1$ in (4.2) represent strong derivatives.

Proof: Write ψ_w' for the weak derivative $(\partial\psi/\partial\lambda)_w$ so that for any $f \in \Phi$, $\partial_\lambda F(\lambda) = \partial_\lambda < f(x), \psi(x,\lambda) >=< f(x), \psi_w'(x,\lambda) >$. Let $B = \{[\psi(x, \lambda + \Delta\lambda) - \psi(x,\lambda)]/\Delta\lambda\} = \{\Delta\psi/\Delta\lambda\}$ for $|\Delta\lambda| < \epsilon$ say. This is a weakly bounded set in Φ' since $\Delta\psi/\Delta\lambda - \psi_w' \to 0$ weakly. Hence B is bounded for the strong topology and evidently $\psi_w' \in \Phi = \Phi''^* = \Phi^*$ (Montel spaces are reflexive). Since ψ_w' is weakly adherent to the bounded set $B \subset \Phi'$ we have $\psi_w' \in \Phi'$. Now use the fact that the weak and strong topologies coincide on $B \cup \psi_w'$ to conclude that $\Delta\psi/\Delta\lambda \to \psi_w'$ strongly. **QED**

Actually for analytic functions one has another recourse based on the Cauchy integral formula. Thus we know, since $F(\lambda) =< f(x), \psi(x,\lambda) >\in \mathcal{H}$,

$$< f(x), \psi(x,\lambda) >= \frac{1}{2\pi i} \oint_C \frac{< f(x), \psi(x,\lambda) >}{\zeta - \lambda} d\zeta =< f(x), \frac{1}{2\pi i} \oint_C \frac{\psi(x,\lambda)}{\zeta - \lambda} d\zeta \quad (4.4)$$

(cf. [8;16] for vector valued integration - here C is e.g. a circle in $\mathbf{C}$ around λ). Assume Φ is reflexive now in which case an integral such as $I(x,\lambda) = \frac{1}{2\pi i} \oint_C [\psi(x,\lambda)/(\zeta - \lambda)] d\zeta$, with e.g. $< f, \psi(x,\lambda) >$ continuous in λ, will belong to $\Phi''^* = \Phi^*$. If Φ' is complete or quasi-complete for example then in fact $I(x,\lambda) \in \Phi'$. Then for $f \in B$ bounded in

Φ one obtains the equation

$$< f, \frac{\Delta I}{\Delta\lambda} - I_\lambda >=< f, \frac{1}{2\pi i}\oint_C \psi(x,\lambda)[\frac{1}{(\zeta-\lambda)(\zeta-\lambda-\Delta\lambda)} - \frac{1}{(\zeta-\lambda)^2}]d\zeta > \quad (4.5)$$

Now $\zeta \to \psi(x,\lambda)$ is weakly continuous so $J = \{\psi(x,\zeta),\ |\zeta| = 1\}$ is weakly compact, hence weakly bounded. If Φ is barreled this means J is strongly bounded and (4.5) shows $(\Delta I/\Delta\lambda) - I_\lambda \to 0$ strongly in Φ'. Hence, since reflexive implies barreled and barreled spaces have quasicomplete duals (cf. [9]) we have

COROLLARY 4.3 In addition to the hypotheses of Theorem 4.1 assume the Φ_i are reflexive. Then the ψ_i are strongly analytic and the derivatives in (4.2) represent strong derivatives.

Next we assume $\mathbf{Q}$ is continuous $\mathcal{H} \to \mathcal{H}$ and $H_1 = H_2 = H$ with $\Phi_i \subset H$ being reflexive and $\Phi_i \subset W_i$ so that Theorem 4.1 holds with Corollary 4.3. Further by Theorem 3.1 for $f \in D(Q_2)$ one has $VQ_2f = Q_1Vf$. A question now arises about the relation of Paley-Wiener properties **PWP** relative to Q_1 and Q_2. We recall that in the model situation of [11;12;28] for example the factorization $\mathcal{P}B^* = \mathcal{Q}$ could be used. Thus let us say that an operator P as above (modeled on second order differential operators Q) has the **PWP** if (thinking of halfline problems on $[0.\infty)$ for convenience of comparison to the examples of section 2), **PWP** : $supp(f) \subset [0,\sigma] \Leftrightarrow \mathcal{P}f = \hat{f}$ is even and entire in $k = \sqrt{\lambda}$ of exponential type σ (i.e. $|\hat{f}(\lambda)| \leq cexp(\sigma|Im(k)|)$). Let $\mathcal{E}_\sigma$ be this space of $\hat{f}$. In such an event, in the examples of section 2 the kernel $B:\ P \to Q$ is easily shown to be lower triangular ($Q \sim -D^2 + q$) so that B^* is given as in (2.3). Then $supp(f) \subset [0,\sigma] \Rightarrow supp(B^*f) \subset [0,\sigma]$ and $\mathcal{P}B^*f = \mathcal{Q}f \in \mathcal{E}_\sigma$. Thus $\Rightarrow$ in **PWP** is transported from P to Q. Let us now examine this in the present context. We remark that one could model our operators on differential operators of order n by working with matrix differential operators but there is also another recourse indicated in [5]. Thus one can often rescale an operator Q by a suitable operator function T(Q) where T(x) is to be defined by the requirements of the situation. The spectral measure $\Gamma(\lambda)$ becomes $\Gamma(T(\lambda))$ and $\psi(x,\lambda) \to \psi(x, T^{-1}(\lambda))$. This amounts to working with $T^{-1}(\lambda)$ entire functions when discussing analyticity so an entire $\hat{f}(\lambda) =< f(x), \psi(x,\lambda) >= \sum c_n\lambda^n \to \sum c_n[T^{-1}(\lambda)]^{-n}$. Thus for $T(x) = x^2,\ Cos(kx) \to Cos(\sqrt{\lambda}x,\ k \in [0,\infty) \cup [0, i\infty) \to \lambda \in (-\infty,\infty)$, etc. One could then refer **PWP** hypotheses to some generic λ etc. but we will keep the k framework here for comparison to section 2.

Thus assume Q_i as indicated and assume Q_1 has **PWP** (recall however that our technique recovers $V \sim B^*:\ Q_2 \to Q_1$ instead of $B:\ Q_1 \to Q_2$ directly). One obtains from (3.9) $\mathcal{Q}_2f = \mathbf{Q}\mathcal{Q}_1f = \mathcal{Q}_1(Vf)$ and we assume $\mathbf{Q}:\ \mathcal{E}_\sigma \to \mathcal{E}_\sigma$ algebraically. Then $\mathcal{Q}_1f$ even entire of exponential type σ implies $\mathcal{Q}_2f$ has the same property along

with $\mathcal{Q}_1(Vf)$. Hence $supp(Vf) \subset [0,\sigma]$ (which is consistent with our identification $V \sim B^*$ and (2.5)). Let us (via experience in section 2) suppose V is an operator $Vf(x) = \int_0^\infty V(x,y)f(y)dy$ with say V(x,y) continuous in y for $y \neq x$ and deduce upper triangularity. Thus (cf. [10]) let $supp(f) \subset [0,\sigma] \Rightarrow supp(\int_0^\infty V(x,y)f(y)dy \subset [0,\sigma]$. Let $f(y) = \delta_n(y-y_0) \in C_0^\infty(0,\infty) \to \delta(y-y_0)$ so $Vf(x) \to V(x,y_0)$ (assume such δ_n belong to $D(Q_1)$ - via our discussion of W^Ω etc. this will pose no problem). For $0 < y_0$, $supp(f) \subset [0, y_0+\epsilon]$ and we obtain $V(x,y_0) = 0$ if $x > y_0 + \epsilon$. Thus $V(x,y) = 0$ for $x > y$ and

$$Vf(x) = \int_x^\infty V(x,y)f(y)dy \tag{4.6}$$

We have shown heuristically

PROPOSITION 4.4. Take $H_1 = H_2 = H = L^2(dx)$. Assume Q_1 has **PWP** and $\mathbf{Q} : \mathcal{E}_\sigma \to \mathcal{E}_\sigma$ algebraically (no continuity is assumed) so $supp(f) \subset [0,\sigma] \Rightarrow \mathcal{Q}_2 f \in \mathcal{E}_\sigma$. Let V defined by the diagram after (3.6), $V = \mathcal{Q}_1^{-1}\mathcal{Q}_2$, be an operator $Vf(x) = \int_0^\infty V(x,y)f(y)dy$ with kernel continuous in y for $y \neq x$ and assume $D(Q_1)$ contains functions $\delta_n(y-y_0) \in C_0^\infty$. Then V has the form (4.6).

We observe that other types of V are possible (e.g. $f \to f$). Now what about the converse, namely $\mathcal{Q}_2 f \in \mathcal{E}_\sigma \Rightarrow supp(f) \subset [0,\sigma]$? Naively one expects an operator V^{-1} to be upper triangular of the same type as V, given that V^{-1} exists. This seems to be a difficult question however and the constructions in [11;12;23;25;26;28] are based on completeness theorems etc. for the generalized eigenfunctions. We will turn to this matter now. The condition **WLP** (weak local property): $supp(f) \subset [0,\sigma] \Leftrightarrow supp(Vf) \subset [0,\sigma]$ can be used to describe this situation. Thus **WLP** $\sim$ V upper triangular $\Leftrightarrow V^{-1}$ upper triangular, but the idea is more general and one can shortcut the development of eigenfunction machinery and achieve on the face of it greater generality by using WLP as a hypothesis in various theorems. For example from what we have stated there follows.

THEOREM 4.5: Assume Q_1 has **PWP** and $\mathbf{Q} : \mathcal{E}_\sigma \to \mathcal{E}_\sigma$ algebraically (no continuity involved) with $H_1 = H_2 = H = L^2$. Then $supp(f) \subset [0,\sigma] \Rightarrow supp(Vf) \subset [0,\sigma]$ and $\mathcal{Q}_2 f \in \mathcal{E}_\sigma$. Further if V has **WLP** then $\mathcal{Q}_2 f \in \mathcal{E}_\sigma \Rightarrow supp(Vf) \subset [0,\sigma] \Rightarrow supp(f) \subset [0,\sigma]$ so Q_2 has **PWP**. On the other hand suppose Q_1 and Q_2 have **PWP**. Then immediately from the diagram after (3.6), $supp(f) \subset [0,\sigma] \Rightarrow supp(Vf) \subset [0,\sigma]$ and $supp(Vf) \subset [0,\sigma] \Rightarrow supp(f) \subset [0,\sigma]$ so V has **WLP**.

Thus let us look now at $\mathcal{Q}_2 f = \mathcal{Q}_1(Vf)$ $(H_1 = H_2 = H = L^2)$ and try to express the **WLP** in terms of generalized eigenfunctions. Written out this says formally, for

suitable f (cf. (4.3))

$$\int f(x)\psi_2(x,\lambda)dx = \int (Vf)(x)\psi_1(x,\lambda)dx = \int f(x)V^*\psi_1(x,\lambda)dx \tag{4.7}$$

Here one imagines e.g. V as an integral operator such as (4.6) (or better (2.3)) with V^* the L^2 adjoint which is presumed to be able to act on $\psi_1 \in \Phi_1'$. Now one knows that in examples from [11;12;23;25;26;28] based on Cos(kx) etc. the generalized eigenfunctions $\psi_i(x,\lambda)$ will be often C^0, C^2, or C^∞ in x and analytic in λ (so the hypotheses on **Q** in Theorem 4.1 are not a priori unreasonable). Moreover $V^* \sim B^{**} = B$ should have the form (via (2.3))

$$V^*f(y) = f(y) + \int_0^y V(x,y)f(x)dx \tag{4.8}$$

so it is not unrealistic to expect (4.7) with some interpretation to yield

$$\psi_2(x,\lambda) = (V^*\psi_1)(x,\lambda) = \psi_1(x,\lambda) + \int_0^x V(y,x)\psi_1(y,\lambda)dy \tag{4.9}$$

This is the natural context then and we want to see how much can be deduced in the more abstract situations.

From [19] (cf. also [4;18]) we have formally for $h \in H = L^2$

$$h(x) = \int (\psi(y,\lambda), h(y))\psi(x,\lambda)d\Gamma; \tag{4.10}$$

$$\|h\|^2 = \int |(\psi,h)|^2 d\Gamma; (\psi(y,\lambda),h(y)) = 0 \ \forall\lambda \Rightarrow h = 0$$

From this one writes formally

$$\delta(x-y) = \int \psi(x,\lambda)\psi(y,\lambda)d\Gamma \tag{4.11}$$

whose meaning is specified by (4.10). Further we can formally represent kernels as in section 2 via

$$\beta(y,x) = kerB = \int \psi_1(x,\lambda)\psi_2(y,\lambda)d\Gamma_1; \tag{4.12}$$

$$\gamma(x,y) = ker\mathcal{B} = \int \psi_1(x,\lambda)\psi_2(y,\lambda)d\Gamma_2$$

In classical situations these are integrals such as $\frac{2}{\pi}\int_0^\infty Cos(\lambda x)Cos(\lambda y)d\lambda = \delta_+(x-y)$ (half line delta function - cf. [11;12]) which are not strictly well defined integrals but

acquire a meaning via distribution theory. For singular differential operators one can find many formulas of the form (4.12) in [11;12] with standard special functions for the ψ_i where everything makes good sense via distribution theory. If formulas such as (4.10) for example cause anxiety one can think of approximating $h \in H$ by $\phi_n \in \Phi$ and/or realize that $h(x) = \int (\psi(y,\lambda), h(y))\psi(x,\lambda)d\Gamma$ is simply another way of saying $h = \mathcal{Q}^{-1}\mathcal{Q}h$. Similarly (4.12) says e.g. (cf. (2.8), (2.10))

$$Bf(y) = < \beta(y,x), f(x) > = \int \mathcal{Q}_1 f \psi_2(y,\lambda)d\Gamma_1 = \tilde{\mathcal{Q}}_2\mathcal{Q}_1 f(y);$$

$$\tilde{\mathcal{Q}}_2 F(y) = \int F(\lambda)\psi_2(y,\lambda)d\Gamma_1 \tag{4.13}$$

We see formally that γ in (4.12) represents an inverse kernel as follows. First for simplicity assume the $d\Gamma_i$ are absolutely continuous with $d\Gamma_i = \gamma_i(\lambda)d\lambda$. Then to go with (4.11) one should have formally

$$\int \psi(x,\lambda)\psi(x,\mu)dx = \delta(\lambda-\mu)/\gamma(\mu) \tag{4.14}$$

This is equivalent formally to

$$F(\lambda) = \int [\delta(\lambda-\mu)/\gamma(\mu)]F(\mu)d\Gamma(\mu) = \tag{4.15}$$

$$= \int \psi(x,\lambda)(\int \psi(x,\mu)F(\mu)d\Gamma(\mu))dx = \mathcal{Q}\mathcal{Q}^{-1}F(\lambda)$$

just as (4.11) is formally equivalent to

$$f(x) = \int \delta(x-y)f(y)dy = \tag{4.16}$$

$$= \int \psi(x,\lambda)(\int \psi(y,\lambda)f(y)dy)d\Gamma(\lambda) = \mathcal{Q}^{-1}\mathcal{Q}f$$

Given the formal structure indicated one has e.g.

$$< \gamma(x,y), \beta(y,\xi) > = \int \psi_1(x,\lambda)(\int \psi_2(y,\mu)dy)\psi_1(\xi,\mu)d\Gamma_2(\lambda)d\Gamma_1(\mu) = \tag{4.17}$$

$$= \int \psi_1(x,\lambda)\frac{\delta(\lambda-\mu)}{\gamma_2(\mu)}\psi_1(\xi,\mu)\gamma_2(\lambda)d\lambda\gamma_1(\mu)d\mu = \int \psi_1(x,\lambda)\psi_1(\xi,\lambda)\gamma_1(\lambda)d\lambda = \delta(x-\xi)$$

so the inversion kernels as in (4.12) are natural.

Now triangularity in the classical examples is proved via hyperbolic PDE or via

analyticity properties of the generalized eigenfunctions (cf. [11;12]). Further the lower triangularity of B and $\mathcal{B}$ is equivalent to the **WLP** for $B^* \sim V$ for the classical examples. However both the contour integration technique from [11;12] or PDE techniques as in [11;12;23;25;26;28] require more detailed knowledge of either the ψ_i or the Q_i. Consider now as a prototypical Q the operator $Q_1 = -D^2$ in $L^2(0,\infty)$ with the generalized eigenfunctions $\psi_1(x,\lambda) = Cos(kx)$ $(\lambda = k^2)$. Let Q_2 have **PWP** (as is common in examples) and look again at Theorem 4.1 (plus the corollaries). Suppose $\mathbf{Q} : \mathcal{H} \to \mathcal{H}$ is continuous in the ucc topology, leading to (4.2). This would imply formally $(\partial_\lambda = \frac{1}{2k}\partial_k)$

$$\psi_2(x,\lambda) \sim \sum_0^\infty a_n(k^2)(\frac{\partial_k}{2k})^n Cos(kx) = \tag{4.18}$$

$$= \sum_0^\infty a_n(k^2)[P_n(x,\frac{1}{k})Cos(kx) + \hat{P}_n(x,\frac{1}{k})Sin(kx)]$$

Thus formally at least ψ_2 would be analytic in x which is unlikely with examples like $Q_2 = -D^2 + q(x)$ with q only continuous. Hence $\mathbf{Q}$ cannot be continuous as indicated in general and we emphasize this via

PROPOSITION 4.6. The hypothesis $\mathbf{Q} : \mathcal{H} \to \mathcal{H}$ continuous in the ucc topology in Theorem 4.1 is usually too strong.

Let us see if Q_1 and Q_2 close in some sense will imply $\mathbf{Q}$ continuous in some sense (perhaps not $\mathcal{H} \to \mathcal{H}$ in ucc topology but in a topology one can adapt to the Levitan theorem). Note that even though $Q\psi = \lambda\psi,\ \psi \in \Phi'$, this only holds for $\lambda \in \sigma_Q \subset \mathbf{R}$ and one cannot directly generate an analyticity argument in λ. Hence we will have to assume $\psi_i(x,\lambda) \in \mathcal{H}$ in λ for the next result. First in order to compare Q_1 and Q_2 let assume $H_1 = H_2 = H$ and $\Phi = \Phi_1 \cap \Phi_2$ is dense in Φ_i. Put on Φ the topology of simultaneous convergence in Φ_1 and Φ_2 so $\Phi \subset \Phi_i \subset H \subset \Phi_i' \subset \Phi'$. Then Φ' can be used as an identification space for the ψ_i. Also $\lambda \to \psi_i(x,\lambda)$ entire with values in Φ_i' will imply these functions are entire with values in Φ'. Further without great loss of generality one could assume also e.g. that $\mathcal{Q}_i^{-1}Z = \Phi_i$ (see remarks before Theorem 4.1). Next we consider the possibility of $\mathbf{Q} : Z \to \mathcal{H}$ being continuous (the reason for this will emerge in the proof of Theorem 4.7 to follow). Recall that $F_n \to 0$ in $Z \sim \|F_n\|_m^Z \to 0$ where $\|F\|_m^Z = sup|F|(1+|\lambda|)^m exp(-a|Im\lambda|)$ (adjust here $\lambda \sim k^2$ as needed for Q modeled on D^2 etc. - cf. remarks before Proposition 4.4). Now locally we can still write $\mathbf{Q}F(z) = (1/2\pi i)\oint_C F(\xi)\mathbf{Q}(\frac{1}{\xi - z})d\xi$ for $z \in K \subset C \subset \hat{K} \subset \Omega$ as before and $F_n \to 0$ in Z implies $F_n \to 0$ in $\mathcal{H}$ so our previous discussion applies. The only additional feature is that $sup|F_n|(1+|\lambda|)^m \le k_m$ for $\lambda \in \mathbf{R}$ and any m, so

if $d\Gamma_1 = \gamma_1 d\lambda$ with $\gamma_1 = O(|\lambda|^p)$ for some p, then

$$\|F_n\|^2_{L^2(d\Gamma_1)} = \int |F_n|^2 \frac{|\lambda^p}{1+|\lambda|)^{2m}}(1+|\lambda|)^{2m} d\lambda \leq \tag{4.19}$$

$$\int \frac{|\lambda|^p}{1+|\lambda|)^{2m}}(|F_n|(1+|\lambda|)^m)^2 d\lambda \leq k_m^2 \|F_n\|_m^Z \ \ (2m \geq p+2)$$

THEOREM 4.7 Assume $H_1 = H_2 = H = L^2(dx)$ for convenience, $\Phi = \mathcal{Q}_1^{-1} Z \cap \mathcal{Q}_2^{-1} Z = \Phi_1 \cap \Phi_2$ is dense in Φ_i, and in H, and that $\psi_i(x,\lambda) \in \Phi'$ are entire in λ. Set $r(x,\lambda) = \psi_2(x,\lambda) - \psi_1(x,\lambda)$ in Φ' with $\hat{r}_1(\nu,\lambda) = \int r(x,\lambda)\psi_1(x,\nu)dx$. Assume (A) For K compact there exists $g(\nu, K) \geq 0 \in L^2(d\Gamma_1(\nu))$ such that $|\hat{r}_1(\nu,\lambda)| \leq g(\nu, K)$ for $\lambda \in K$ (B) $d\Gamma_1$ is absolutely continuous with $d\Gamma_1 = \gamma_1 d\lambda$ and $|\gamma_1| = O(|\lambda|^p)$ (C) $\mathbf{Q} : Z \to \mathcal{H}$ is continuous. Then (4.2) holds locally in Φ'.

Proof: Clearly $r(x,\lambda)$ is entire in λ with values in Φ' and via Parseval for $\mathcal{Q}_1$ one has

$$\mathbf{Q}\hat{f}_1(\lambda) = \hat{f}_2(\lambda) = \hat{f}_1(\lambda) + \int \hat{r}_1(\nu,\lambda)\hat{f}_1(\nu)d\Gamma_1(\nu) \tag{4.20}$$

(i.e. $\int f(x)r(x,\lambda)dx = \int \hat{f}_1\hat{r}_1(\nu,\lambda)d\Gamma_1(\nu)$). This is an integral equation of Carleman type and (A) ensures that one can use dominated convergence ideas in the integration. In particular $\hat{f}_2(\lambda)$ and $\hat{f}_1(\lambda)$ are entire and we will show that $\hat{f}_n^1 \xrightarrow{Z} 0 \Rightarrow \mathbf{Q}\hat{f}_n^1 = \hat{f}_n^2 \xrightarrow{ucc} 0$. One need only estimate the integral term $J(\lambda) = \int \hat{r}_1(\nu,\lambda)\hat{f}_1(\nu)d\Gamma_1(\nu)$. First for $f_n \in \Phi$ as above, one has $|\hat{f}_n^1(\nu)| \leq k_m(1+|\nu|)^{-m}$ for any m as above and we choose m so that $\|\hat{f}_n^1\|_{L^2(d\Gamma_1} \leq k_m$. Then consider for $\hat{f}_n^1 \in Z$

$$sup_{\lambda\in K}|J_n(\lambda)| \leq \int g(\nu,K)|\hat{f}_n^1(\nu)|d\Gamma_1(\nu) \tag{4.21}$$

$$\leq (\int g^2 d\Gamma_1)^{\frac{1}{2}}(\int |\hat{f}_n^1(\nu)|^2 d\Gamma_1)^{\frac{1}{2}} \leq G k_m \|\hat{f}_n^1\|_m \ (2m \geq p+2)$$

But $\hat{f}_n^1 \xrightarrow{Z} 0$ means $\|\hat{f}_n^1\|_m \to 0$ for any m, hence in particular for $2m \geq p+2$, $J_n(\lambda) \to 0$ in $\mathcal{H}$. Consequently $\mathbf{Q}\hat{f}_1 = \hat{f}_2$ can be written as in (4.1) for $f \in \Phi$. This leads to (4.3) and the identification as in (4.2) with the ψ_i as transform objects, or locally in Φ' strongly. **QED**

REMARK 4.8: The proof holds for any W^Ω type space with adjustment of hypotheses on Φ_i (i.e. $\Phi_i = \mathcal{Q}_i^{-1}W^\Omega$, etc.) but the strongest version would involve only $\mathbf{Q} : \mathcal{H} \to \mathcal{H}$ algebraically and $\mathbf{Q}F_n \to 0$ in $\mathcal{H}$ provided $F_n \to 0$ in $\mathcal{H}$ plus $(1+|\lambda|)^m sup_{\lambda\in\mathbf{R}}|F_n(\lambda)| \to 0$ for some $2m \geq p+2$. Of course other hypotheses on $|\hat{r}_1(\nu,\lambda)|$ could also be made. Note that given Φ as indicated in Theorem 4.7, (4.18)

would suggest that for $\psi_1 \sim Cos(kx)$, which is analytic in x, if Q_2 is suffficiently close to $Q_1 = -D^2$ (measured by $\hat{r}_1$), then $\psi_2(x,\lambda)$ might also be analytic in x, via continuity of $\mathbf{Q}$ in some sense. Further investigation of such matters is clearly indicated.

REMARK 4.9. Given $Q_1 = -D^2$ and $Q_2 = -D^2 + q(x)$ one has formally $-\psi_2'' + q\psi_2 = \lambda\psi_2 = k^2\psi_2$. Given a transmutation B as in (2.2) with $\psi_2(x,\lambda) = Cos(kx) + \int_0^x K(x,t)Cos(kt)dt$ one will have ψ_2 analytic in λ or k. Further from the differential equation, if q(x) is analytic in x one expects ψ_2 will be analytic in x. Thus such a situation should involve formulas of the type (4.18). In other words analytic q(x) could produce closeness of Q_2 and Q_1 in the sense of Theorem 4.7. One suspects also that relations could be established here to the results of [27].

REMARK 4.10. We note here another approach to determining transmutations analogous to the diagram after (3.6). Thus $\mathcal{B} : Q_2 \to Q_1$ can be formally defined as $\mathcal{B} = \tilde{\mathcal{Q}}_1\mathcal{Q}_2$ (cf. (2.11), (4.13)), where $\tilde{\mathcal{Q}}_1 F(x) = \int F(\lambda)\psi_1(x,\lambda)d\Gamma_2$. Consider for simplicity a situation where say $Z \subset L^2(d\Gamma_1) \cap L^2(d\Gamma_2)$ is dense in each L^2 and $\Phi_i = \mathcal{Q}_i^{-1}Z$ with $\Phi = \Phi_1 \cap \Phi_2$ dense in $H = H_1 = H_2$ and in $D(Q_i)$. Then look at a diagram ($d\Gamma_i = \gamma_i d\lambda$) with $\mathbf{P} = \gamma_2/\gamma_1$

$$\begin{array}{ccc} \hat{f}_2 & \xrightarrow{\mathbf{P}} & \mathbf{P}\hat{f}_2 \\ \mathcal{Q}_2 \uparrow & \searrow \tilde{\mathcal{Q}}_1 & \downarrow \mathcal{Q}_1^{-1} \\ f \in \Phi & \xrightarrow{\mathcal{B}} & \mathcal{B}f \in H \end{array}$$

and observe here that for $\mathbf{P} = \gamma_2/\gamma_1$

$$\int \hat{f}_2\psi_1 d\Gamma_2 = \int \hat{f}_2\frac{\gamma_2}{\gamma_1}\psi_1 d\Gamma_1; \tilde{\mathcal{Q}}_1 = \mathcal{Q}_1^{-1}\mathbf{P}; \mathcal{B} = \tilde{\mathcal{Q}}_1\mathcal{Q}_2 \tag{4.22}$$

We record here in this connection

THEOREM 4.11. For $\mathbf{P} = \sqrt{\gamma_2/\gamma_1}$ the diagram (with $\tilde{\mathcal{Q}}_1$ removed) defines a new kind of transmutation $\mathcal{B} : Q_2 \to Q_1$, satisfying $Q_1\mathcal{B}f = \mathcal{B}Q_2 f$ for $f \in D(Q_2)$. For $\mathbf{P} = \gamma_2/\gamma_1$ with $|\gamma_2/\gamma_1| \leq c$ (and $\tilde{\mathcal{Q}}_1$) we get $\mathcal{B} \sim B^{-1}$ (cf. (2.11) and (4.22)).

Proof: Take $f \in D(Q_2)$ so that

$$\lambda\mathcal{Q}_1(\mathcal{B}f) = \mathbf{P}\lambda\mathcal{Q}_2(f) = \mathbf{P}\mathcal{Q}_2(Q_2 f) = \mathcal{Q}_1(\mathcal{B}Q_2 f) \tag{4.23}$$

(cf. here (3.8)-(3.11) - the conditions on $\mathbf{P}$ insure that everything makes sense). Then we want to show $(\mathcal{B}Q_2 f, h) = (\mathcal{B}f, Q_1 h)$ for $h \in D(Q_1)$ which would imply that

$\mathcal{B}f \in D(Q_1)$ with $\mathcal{B}Q_2 f = Q_1\mathcal{B}f$. Thus represent h as $h = \mathcal{Q}_1^{-1}\mathcal{Q}_1 h = \int(\mathcal{Q}_1 h)\psi_1\gamma_1 d\lambda$ and one has $Q_1 h = \int(\mathcal{Q}_1 h)\lambda\psi_1\gamma_1 d\lambda$ with

$$(Q_1 h, \mathcal{B}f) = \int(\mathcal{Q}_1 h)\gamma_1\lambda(\psi_1, \mathcal{B}f)d\lambda = \tag{4.24}$$

$$= \int(\mathcal{Q}_1 h)\gamma_1\lambda\mathcal{Q}_1(\mathcal{B}f)d\lambda = \int(\mathcal{Q}_1 h)\gamma_1\mathcal{Q}_1(\mathcal{B}Q_2 f)d\lambda = (h, \mathcal{B}Q_2 f)$$

(the last equation by Parseval). **QED**

This shows that $\mathcal{B} :\ Q_2 \to Q_1$ is a transmutation and we have developed constructions of $\mathcal{B} \sim B^{-1}$ and $V \sim B^*$. Note also that if **P** is a polynomial for example then **P** maps $\mathcal{H} \to \mathcal{H}$ or $Z \to Z$. One can surely enhance the investigation of **PWP** and related matters using Theorem 4.11 with the previous results and we will return to this at another time.

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